https://www.linkedin.com/feed/update/urn:li:activity:6623654257499471872

Let $a,b,c \in \left[\frac{1}{2},1\right]$. Prove that $2 \le \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b} \le 3$.

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1. $\sum \frac{a+b}{1+c} \ge 2 \Leftrightarrow \sum\left(\frac{a+b}{1+c}+1\right) \ge 5 \Leftrightarrow (a+b+c+1)\sum \frac{1}{1+a} \ge 5$.

Since by Cauchy Inequality $\sum \frac{1}{1+a} \ge \frac{9}{3+a+b+c}$ then

$(a+b+c+1)\sum \frac{1}{1+a} \ge \frac{9(a+b+c+1)}{a+b+c+3} = 9\left(1-\frac{2}{a+b+c+3}\right) \ge$
$9\left(1-\frac{2}{3\cdot 1/2+3}\right) = 5$.

2. Let $F(a,b,c) := \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b}$ and $f(x) := F(a,b,x)$.

Since $f''(x) = \frac{2(a+b)}{(x+1)^3} \ge 0$ then $f(x)$ is concave up function and,

therefore, due to symmetry of $F(a,b,c)$, we have

$\max\limits_{a,b,c\in[1/2,1]} F(a,b,c) = \max\{F(1,1,1),F(1,1,1/2),F(1,1/2,1/2),F(1/2,1/2,1/2)\} = F(1,1,1) = 3$.